Moment evolution and level-crossing statistics in dichotomous and multilevel flows with time-dependent control parameters

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We study the dynamics of the first two moments and of threshold crossings by the stochastic trajectory in dichotomous diffusion $\dot{x}=\xi(t)$, where $\xi(t)$ is a dichotomous Markov process. The transition rate of the latter is regarded as a control parameter and allowed to have specified time variations. The stabilizing or destabilizing effect of this variation is demonstrated, and qualitative changes in the statistical properties of the system are shown to occur. The analysis is then extended to linear dichotomous flow, and to a generalization of dichotomous diffusion in which x is driven by a multilevel Markov noise.

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I. INTRODUCTION

One of the most useful indicators of the stability of a natural phenomenon is the smallness of the standard deviation of the relevant variable(s) (and its higher order analogs), as compared to the mean [1,2]. Less familiar, although quite relevant, is the sparseness of crossings by the dynamical trajectory of levels distinctly different from the mean (or, more significantly, from the most probable value), the statistics of such *level crossings* being in turn intimately related to the problem of *extreme values* [3–5].

Investigations of moment dynamics and level crossing or extreme value statistics are usually limited to *stationary* dynamical systems, i.e., systems that are subjected to fixed control parameters and possess sufficiently strong ergodic properties. There are, however, compelling reasons for extending these investigations to systems forced by time-dependent control parameters. Two particularly significant instances calling for such an extension are possible changes of trends in atmospheric dynamics in connection with anthropogenic forcings [6], and the dynamics of financial markets in which background information is continuously updated by the outcome of the dynamics itself [7]. Other systems in which similar effects come into play are switching phenomena in electronic or optical devices, and chemical or biological processes under real-world environmental conditions. The aim of the present work is to analyze the role of the time dependence of the control parameters on moment evolution and level-crossing statistics in a class of stochastic dynamical systems forced by dichotomous noise [3,4,8–10] and a multilevel generalization thereof [11].

The principal motivation for focusing on dynamical sytems of this type is their genericness. They may indeed be mapped, via appropriately taken limits in their control parameters, onto familiar processes such as ordinary continuous diffusion (the Wiener process) and random walks, while for more typical values of these parameters they are able to account for finite correlation times and memory effects. It is therefore not surprising that they are widely used in the modeling of a variety of processes, from telecommunications (the random telegraph signal) to dipersion in porous media. An additional motivation is the possibility of accounting for the discreteness and non-Gaussian character of the noise source, features that are often overlooked in passing to the familiar limit of Gaussian white noise. Finally, while some work has been reported on the role of time-dependent control parameters in stochastic dynamical systems forced by white noise [12–14], to our knowledge no such studies have been undertaken in the presence of dichotomous-type noise.

In what follows, we first consider the simplest case of dichotomous diffusion. The dynamics of the first two moments and the level crossings are analyzed in Sec. II and Sec. III, respectively, the relevant control parameter being the switching rate λ between the two levels of the noise. The case of a more general dichotomous flow, including an additional drift term, is considered in Sec. IV. Section V is devoted to the extension of some of the results to the case where the system is forced by a multilevel, discrete, exponentially correlated noise. Our main conclusions are summarized in Sec. VI. While we have focused in this work on the switching rate as the time-dependent control parameter, in principle other such parameters could also be considered—for instance, a systematic time dependence of the levels of the noise, different time dependences of the rates of switching between these, and so on.

II. DICHOTOMOUS DIFFUSION: MEAN AND VARIANCE

A. Moment equations

Dichotomous diffusion is a stochastic process continuous in time (t) and state space (x), described by the evolution equation

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$$\dot{x} = \xi(t), \tag{1}$$

where, to avoid inessential complications, we take $\xi(t)$ to be a symmetric dichotomous noise: a Markov process switching between the levels $+c$ and $-c$ at a mean transition rate λ, which will be allowed to vary in a deterministic manner with time. One can write down a master equation governing the probability density of x, by realizing that Eq. (1) describes the motion of a random walker on the real line moving at constant speed c, with instantaneous reversals of the direction of motion at random instants. This leads to

$$\frac{\partial P_+}{\partial t} + c\frac{\partial P_+}{\partial x} = \lambda(t)(P_- - P_+)$$

and

$$\frac{\partial P_-}{\partial t} - c\frac{\partial P_-}{\partial x} = \lambda(t)(P_+ - P_-), \tag{2}$$

$P_+(x,t)$ and $P_-(x,t)$ being the probability densities when the motion is in the sense of increasing x and decreasing x, respectively. The total probability density for the process $x(t)$ is clearly $P(x,t) = P_+(x,t) + P_-(x,t)$. Introducing also the "excess" density $Q = P_+ - P_-$, Eqs. (2) transform to

$$\frac{\partial P}{\partial t} + c\frac{\partial Q}{\partial x} = 0 \quad \text{and} \quad \frac{\partial Q}{\partial t} + c\frac{\partial P}{\partial x} = -2\lambda(t)Q. \tag{3}$$

Eliminating Q from these two relations, one obtains a closed equation for P, namely,

$$\frac{\partial^2 P}{\partial t^2} + 2\lambda(t)\frac{\partial P}{\partial t} - c^2\frac{\partial^2 P}{\partial x^2} = 0. \tag{4}$$

For definiteness, throughout this work (except where specified otherwise) we shall impose initial conditions corresponding to a walker initially located at $x = x_0$ and moving in the direction of increasing x [i.e., $\xi(0) = +c$], so that

$$P(x,0) = \delta(x - x_0) \quad \text{and} \quad [\partial P/\partial t]_{t=0} = -c\,\delta'(x - x_0). \tag{5}$$

An incidental advantage of these asymmetric initial conditions is that one can now clearly see the extent to which the memory of the initial state is retained in the asymptotic expressions for various quantities, an aspect which is obscured if we choose, say, symmetric initial conditions such as $P(x,0) = \delta(x)$, $\dot{P}(x,0) = 0$.

Multiplying Eq. (4) in succession by x and x^2, integrating over x, and using no-flux boundary conditions at $x = \pm\infty$, one finds the following equations for the moments $m_1 = \langle x(t) \rangle$ and $m_2 = \langle x^2(t) \rangle$:

$$\frac{d^2 m_1}{dt^2} + 2\lambda(t)\frac{dm_1}{dt} = 0, \tag{6}$$

and

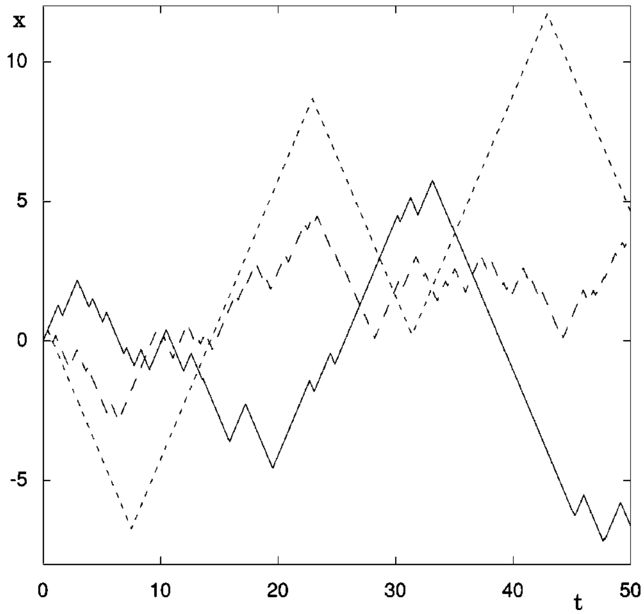

FIG. 1. Typical time evolution of a dichotomous diffusion process [Eq. (1)], with $\lambda = \lambda_0 + \epsilon t$, $\epsilon = 0$ (full line), $\epsilon = 0.05$ (dashed line), and $\lambda = \lambda_0(1 + t/\tau)^{-1}$ (dotted line). Parameter values $c = \lambda_0 = \tau = 1$, and initial conditions $x_0 = 0, \xi_0 = +c$. The units of x and t in this figure and in the subsequent ones are fixed by the choice of parameter values $c = 1$ and $\lambda_0 = 1$.

$$\frac{d^2 m_2}{dt^2} + 2\lambda(t)\frac{dm_2}{dt} - 2c^2 = 0. \tag{7}$$

The initial conditions satisfied by these moments follow from those in Eq. (5), and are given by

$$m_1(0) = x_0, \quad \dot{m}_1(0) = c,$$

$$m_2(0) = x_0^2, \quad \dot{m}_2(0) = 2cx_0. \tag{8}$$

Clearly, the behavior of the probability density and its moments depends crucially on the control parameter $\lambda(t)$. We shall be interested in situations in which $\lambda(t)$ increases or decreases monotonically with time [Eqs. (15) and (25) below], and in comparing these with the "reference case" of constant λ. Figure 1 depicts typical realizations of the dichotomous diffusion process in these three situations, and gives a very preliminary idea of what one might expect. A more quantitative analysis is carried out in the subsections that follow.

B. Time-independent λ

For a constant value of λ, Eqs. (6) and (7) can be integrated in a straightforward manner. The solutions obtained for $m_1(t)$ and $m_2(t)$ subject to the initial conditions (8) are

$$m_1(t) = x_0 + \frac{c}{2\lambda}(1 - e^{-2\lambda t}) \tag{9}$$

and

$$m_2(t)=x_0^2+\frac{c^2t}{\lambda}+\frac{c}{\lambda}\left(x_0-\frac{c}{2\lambda}\right)(1-e^{-2\lambda t}). \tag{10}$$

The mean displacement saturates to $x_0+c/2\lambda$, the specific value reflecting the initial conditions chosen (in both position and velocity) and the fact that there is no stable drift term in Eq. (1). In contrast, the second moment exhibits, after a transient period of the order of $(2\lambda)^{-1}$, diffusive behavior—a consequence, ultimately, of the central limit theorem—with an effective diffusion coefficient equal to $c^2/2\lambda$. This can be seen more clearly from the expression for the variance $\delta m_2 =\langle x^2\rangle-\langle x(t)\rangle^2$, which is

$$\delta m_2(t)=\frac{c^2}{2\lambda^2}\left[2\lambda t-(1-e^{-2\lambda t})-\frac{1}{2}(1-e^{-2\lambda t})^2\right]. \tag{11}$$

This is also consistent with the well-known fact that dichotomous diffusion reduces to the usual diffusion (Wiener) process in the limit in which both c and λ tend to infinity such that c^2/λ remains finite, with a diffusion constant that is given precisely by $D=\lim c^2/(2\lambda)$.

C. Time-dependent λ

Equations (6) and (7) can again be integrated subject to the initial conditions (8). We find

$$m_1(t)=x_0+c\int_0^t dt'\,e^{-2\int_0^{t'}\lambda(\tau)d\tau} \tag{12}$$

and

$$m_2(t)=x_0^2+2cx_0\int_0^t dt'\,e^{-2\int_0^{t'}\lambda(\tau)d\tau}+2c^2\int_0^t dt'\,e^{-2\int_0^{t'}\lambda(\tau)d\tau}\int_0^{t'}dt''e^{2\int_0^{t''}\lambda(\tau)d\tau}. \tag{13}$$

Correspondingly, the variance is given by

$$\delta m_2(t)=2c^2\left[\int_0^t dt'\,e^{-2\int_0^{t'}\lambda(\tau)d\tau}\int_0^{t'}dt''e^{2\int_0^{t''}\lambda(\tau)d\tau}-\frac{1}{2}\left(\int_0^t dt'\,e^{-2\int_0^{t'}\lambda(\tau)d\tau}\right)^2\right]. \tag{14}$$

1. λ linearly increasing with time

We call such a variation a ‘‘ramp,’’ and write it as

$$\lambda(t)=\lambda_0\left(1+\frac{t}{\tau}\right)\equiv\lambda_0+\epsilon t \tag{15}$$

where ϵ expresses the increase of the switching rate with time. The integrals in Eq. (12) can be evaluated in closed form to yield

$$m_1(t)=x_0+\frac{c}{2}\sqrt{\frac{\pi}{\epsilon}}e^{\lambda_0^2/\epsilon}\times\left[\operatorname{erf}\left(t\sqrt{\epsilon}+\frac{\lambda_0}{\sqrt{\epsilon}}\right)-\operatorname{erf}\left(\frac{\lambda_0}{\sqrt{\epsilon}}\right)\right]. \tag{16}$$

We see that, despite the continuous increase of $\lambda(t)$, the first moment attains a limiting value. An asymptotic evaluation for $t\gg1/\sqrt{\epsilon}$ leads to

$$m_1(t)\approx x_0+\frac{c}{2\lambda_0}-\frac{c\epsilon}{4\lambda_0^3}+\cdots. \tag{17}$$

Therefore, to leading order in ϵ, the effect of the ramp is to *depress* the value of m_1 in this asymptotic regime: in a sense, increasingly frequent switching between the two noise levels tends to ‘‘stabilize’’ the mean—a result that could at first sight seem to be counterintuitive. By stabilization we mean here the fact that the drift of the mean from x_0 to $x_0+c/(2\lambda_0)$ is at least partially arrested, for the asymptotic value of the second term in Eq. (16) is still less than $c/(2\lambda_0)$.

As the integrals in Eq. (13) for the second moment cannot be carried out in closed form in the case of a ramp [Eq. (15)], we turn to the numerical simulation of dichotomous diffusion. In doing so, care must be taken to properly incorporate the time dependence of $\lambda(t)$ in the evolution of the stochastic trajectory of the noise $\xi(t)$, paying particular attention to its nonstationary nature. Now, the probability that the noise remains at its initial level until time t (i.e., the ‘‘lifetime’’ distribution for either state of the noise) is given by

$$\pi(t)=\exp\left(-\int_0^t\lambda(t')dt'\right), \tag{18}$$

while the probability of a given transition history is

$$\pi(t_1,t_2,\ldots)=e^{-\int_0^{t_1}\lambda(t')dt'}\lambda(t_1)\,\delta t_1\times e^{-\int_{t_1}^{t_2}\lambda(t')dt'}\lambda(t_2)\,\delta t_2\cdots. \tag{19}$$

If one starts in the state $\xi(0)=+c$, the probability that $\xi(t)=\pm c$ at any subsequent time t is then given by

$$P_\pm(t)=\tfrac{1}{2}\,(1\pm e^{-2\int_0^t\lambda(t')dt'}). \tag{20}$$

With this background, one can simulate the dichotomous noise process in the presence of a ramp by suitably generalizing the method developed by Gillespie [15] in the case of exponentially distributed lifetimes [i.e., $\lambda=$const in Eq. (18)]. The salient results of the simulation, as far as the first moment $m_1(t)$ and the variance $\delta m_2(t)$ are concerned, are summarized in Fig. 2. (Here, and in all the numerical results in Secs. II–IV, the value of λ_0 has been set equal to unity.) In Fig. 2(a) we see, in agreement with the analytic result of Eq. (16), that m_1 indeed approaches an asymptotic level, which moreover decreases with increasing ϵ. Figure 2(b) depicts the behavior of the variance. The main conclusion is that the increase of the variance with time is depressed as ϵ in-

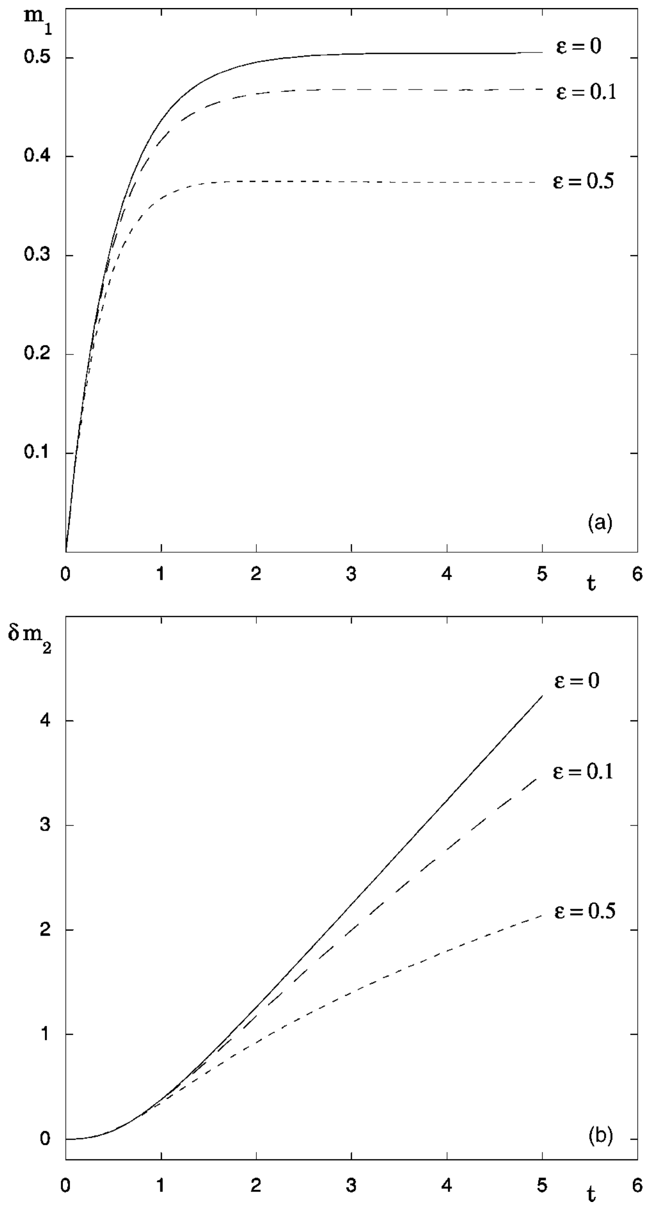

FIG. 2. First moment (a) and second moment (b) for dichotomous diffusion obtained numerically from Eq. (1) after an averaging over 10^6 realizations and with different values of the ramp parameter ϵ [Eq. (15)]. Parameter values and initial conditions as in Fig. 1.

creases. This illustrates further the stabilization of the system under the influence of the ramp, at least as far as the behavior of the moments is concerned.

To derive more quantitative results on the behavior of the second moment, we consider the limit of small ϵ and resort to multiple time scale analysis (or an "adiabatic approximation"). One introduces a slow time variable

$$\tilde{t} = \lambda_0 + \epsilon t \tag{21}$$

[which is essentially $\lambda(t)$ itself for the simple ramp], together with the expansion

$$m_2(\tilde{t}) = \epsilon^{-1} m_2^{(0)}(\tilde{t}) + \cdots . \tag{22}$$

Equation (7) then yields, to dominant order,

$$2\tilde{t}\frac{dm_2^{(0)}}{d\tilde{t}} - 2c^2 = 0, \tag{23}$$

which can be integrated in a straightforward manner to yield, for sufficiently long times,

$$m_2(t) \approx (c^2/\epsilon)\ln(\lambda_0 + \epsilon t), \tag{24}$$

up to an additive constant. We conclude that the diffusive behavior characteristic of the process when λ is fixed is now replaced by a *subdiffusive* one. In Figs. 3(a) and 3(b) we show the fit, by the formula of Eq. (24), to the data of Fig. 2(b) corresponding to $\epsilon = 0.1$ and $\epsilon = 0.5$. The agreement is excellent, confirming the validity of our asymptotic analysis.

2. λ decreasing with time

We express this as

$$\lambda(t) = \lambda_0\left(1 + \frac{t}{\tau}\right)^{-1}. \tag{25}$$

Equation (12) now yields

$$m_1(t) = x_0 + \frac{c\,\tau}{(2\lambda_0\tau - 1)}\left[1 - \left(\frac{\tau}{t+\tau}\right)^{2\lambda_0\tau - 1}\right]. \tag{26}$$

Two qualitatively different cases can be distinguished.

(i) If $2\lambda_0\tau - 1 > 0$, Eq. (26) leads to a well-defined asymptotic limit for the first moment, which we may write as

$$m_1(\infty) = x_0 + \frac{c}{2\lambda_0}\left(\frac{2\lambda_0\tau}{2\lambda_0\tau - 1}\right). \tag{27}$$

Comparison with Eqs. (9) and (17) shows that m_1 is now *enhanced* relative to its value when λ is constant, in contrast to the case of the ramp.

(ii) When $2\lambda_0\tau - 1 < 0$, the first moment becomes unbounded as $t \to \infty$, according to

$$m_1(t) \approx \left(\frac{c\tau}{1 - 2\lambda_0\tau}\right) t^{1 - 2\lambda_0\tau}. \tag{28}$$

These two regimes are separated by the borderline value $2\lambda_0\tau - 1 = 0$, at which

$$m_1(t) = x_0 + c\tau\ln\left(\frac{t+\tau}{\tau}\right), \tag{29}$$

which diverges logarithmically as $t \to \infty$. The destabilization of the system induced by the steady decrease of λ is now manifest.

For the variance, one obtains from Eqs. (14) and (25) the following exact expression (for $2\lambda_0\tau - 1 > 0$ as well as $2\lambda_0\tau - 1 < 0$):

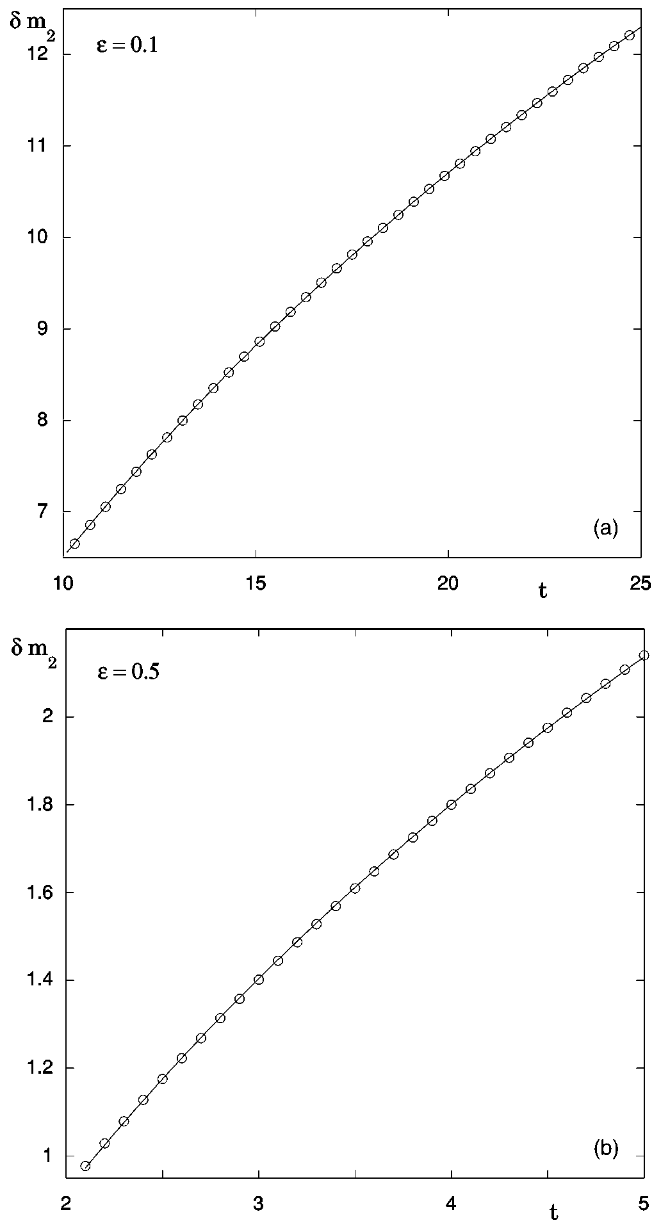

FIG. 3. Time evolution for $t \gg \epsilon^{-1/2}$ of the second moment (empty circles) as obtained numerically from Eq. (1) with (a) $\epsilon = 0.1$ and (b) $\epsilon = 0.5$. The full line depicts the asymptotic analytical expression Eq. (24). Parameter values as in Fig. 2.

$$\delta m_2(t) = \frac{c^2 t(t+2\tau)}{(2\lambda_0\tau+1)} + \frac{2c^2\tau^2}{(4\lambda_0^2\tau^2-1)}[(t/\tau+1)^{1-2\lambda_0\tau}-1] - \frac{c^2\tau^2}{(2\lambda_0\tau-1)^2}[(t/\tau+1)^{1-2\lambda_0\tau}-1]^2. \quad (30)$$

The dominant dependence as $t \to \infty$ is, therefore,

$$\delta m_2(t) \approx \frac{c^2 t^2}{2\lambda_0\tau+1}, \quad (31)$$

implying that the displacement from the mean exhibits *ballistic* behavior—an additional signature of the weakening of the stability of the system induced by the sytematic decrease of $\lambda(t)$ with time.

III. LEVEL CROSSINGS IN DICHOTOMOUS DIFFUSION

A. General formulas

We now turn to the statistics of level crossings in dichotomous diffusion. As a more "local" probe, this is a valuable supplement to the study of the moments in obtaining information on the general stability of the system.

The random variable $x(t)$ defined by Eq. (1) is a continuous process with a bounded velocity. For such a process, it can be shown [4] that the mean value of the number $N(x_{th};0,T)$ of crossings of any specified level or threshold $x = x_{th}$, in the time interval between $t=0$ and $t=T$, is given by

$$\langle N(x_{th};0,T)\rangle = \int_0^T dt \int_{-\infty}^{\infty} d\dot{x}|\dot{x}|P(x_{th},\dot{x},t), \quad (32)$$

where $P(x,\dot{x},t)$ is the joint probability density of the system in phase space. This formula counts both upcrossings $[N_\uparrow(x_{th};0,T)]$ and downcrossings $[N_\downarrow(x_{th};0,T)]$ of x_{th}; for the mean value of the former (latter) alone, the lower (upper) limit of integration over $\dot{x}$ in Eq. (32) must be replaced by 0. The mean squared value of $N(x_{th};0,T)$ involves the two-time probability density in phase space, and is given by

$$\langle N^2(x_{th};0,T)\rangle = \int_0^T dt_1 \int_0^T dt_2 \int_{-\infty}^{\infty} d\dot{x}_1 \int_{-\infty}^{\infty} d\dot{x}_2 \times |\dot{x}_1||\dot{x}_2| P(x_{th},\dot{x}_1,t_1;x_{th},\dot{x}_2,t_2). \quad (33)$$

These general formulas simplify somewhat in the case of dichotomous diffusion. As the speed $|\dot{x}|$ is always equal to c in this instance, we get

$$\langle N(x_{th};0,T)\rangle = c\int_0^T dt P(x_{th},t), \quad (34)$$

where $P(x,t) = P_+(x,t) + P_-(x,t)$ as already defined. Similarly,

$$\langle N^2(x_{th};0,T)\rangle = c^2 \int_0^T dt_1 \int_0^T dt_2 P(x_{th},t_1;x_{th},t_2). \quad (35)$$

Initial conditions must of course be specified. We shall use the same ones as before, i.e., those corresponding to an initial position x_0 and motion in the direction of increasing x, as in Eq. (5).

B. Time-independent λ

To illustrate the essential features of level crossing, it suffices to consider the case $x_0 = 0, \quad x_{th} = 0$. (The modifications introduced when x_0 and x_{th} are arbitrary are discussed subsequently.) Using the known solutions (see, e.g., [10]) for $P_{\pm}(x,t)$ in the case of dichotomous diffusion, we get (for the specific initial conditions we have assumed)

$$\langle N(0;0,T)\rangle = \frac{1}{2}\int_0^{\lambda T} e^{-u}[I_0(u)+I_1(u)]du, \tag{36}$$

where I_n is the modified Bessel function of order n. The two terms in the integrand represent, respectively, the contributions from downcrossings and upcrossings of the zero level.

In order to compute $\langle N^2\rangle$, we need the two-time probability density $P(x_1,t_1;x_2,t_2)$. This can be determined in the present instance by recalling that the *pair* $(x,\dot{x})$ constitutes a two-dimensional Markov process. Using this fact, as well as the solutions [10] for dichotomic diffusion for all four probability densities $P(x,\xi,t|x_0,\xi_0,t_0)$ where $\xi=\pm c$ and $\xi_0=\pm c$, we obtain from Eq. (35) after some simplification the result

$$\langle N^2(0;0,T)\rangle = \frac{1}{2}\int_0^{\lambda T} du_1 e^{-u_1}\int_0^{u_1} du_2 \times[I_0(u_1-u_2)+I_1(u_1-u_2)] \times[I_0(u_2)+I_1(u_2)]. \tag{37}$$

Exploiting the fact that this expression is in the form of a convolution, one of the integrations in Eq. (37) can be carried out. We then get

$$\langle N^2(0;0,T)\rangle = \lambda T - \int_0^{\lambda T} du e^{-u}[I_0(u)+I_1(u)] = \lambda T - 2\langle N(0;0,T)\rangle. \tag{38}$$

When T is very large (i.e., $\lambda T \gg 1$), Eq. (36) yields, for the asymptotic behavior of the mean number of zero crossings, the expression

$$\langle N(0;0,T)\rangle = \left(\frac{2\lambda T}{\pi}\right)^{1/2}\left[1+\frac{1}{8\lambda T}+O((\lambda T)^{-2})\right]. \tag{39}$$

Similarly, for the variance of this number we find the asymptotic expansion

$$\langle \delta N^2(0;0,T)\rangle = \left(1-\frac{2}{\pi}\right)\lambda T - \left(\frac{8\lambda T}{\pi}\right)^{1/2} - \frac{2}{\pi} + O((\lambda T)^{-1/2}). \tag{40}$$

Hence the relative fluctuation in the number of zero crossings is, for $\lambda T \gg 1$,

$$\frac{\langle \delta N^2(0;0,T)\rangle^{1/2}}{\langle N(0;0,T)\rangle} = \left(\frac{\pi}{2}-1\right)^{1/2} - \left(1-\frac{2}{\pi}\right)^{-1/2}(\lambda T)^{-1/2} + \cdots. \tag{41}$$

Thus level crossings in dichotomous diffusion display strong fluctuations, as the standard deviation is comparable to the mean value. Moreover, as $\lambda T \to \infty$, their ratio tends to an absolute constant $(\pi/2-1)^{1/2}$. All these results are fully confirmed by numerical simulations. Figure 4 shows some of the relevant features in this regard. In particular, the full line in Fig. 4(c) represents the exact analytic expression for the relative fluctuation in N, while the crosses represent the results of simulation.

We now comment briefly on level crossings in the case of arbitrary threshold value x_{th}. Starting (for generality) at an arbitrary initial position x_0 as well, we find that the main effect is a time delay $T_d = |x_{th}-x_0|/c$ after which level crossings commence. For $x_{th} > x_0$, we get (when $t > T_d$)

$$\langle N(x_{th};0,T)\rangle = e^{-\lambda T_d} + \frac{1}{2}\int_{\lambda T_d}^{\lambda T} e^{-u}\left\{I_0(w)+\left(\frac{u+\lambda T_d}{u-\lambda T_d}\right)^{1/2} I_1(w)\right\}du, \tag{42}$$

where $w(u,x_{th},x_0)=(u^2-\lambda^2 T_d^2)^{1/2}$. For $x_{th}<x_0$ the first term on the right ($e^{-\lambda T_d}$) is absent, and the prefactor of I_1 in the integrand is $[(u-\lambda T_d)/(u+\lambda T_d)]^{1/2}$.

C. Time-dependent λ

Now consider the effect of a systematic ramplike increase of the switching rate $\lambda(t)$ upon level crossings in dichotomous diffusion. As analytic expressions for the probability densities concerned are not available in this case, numerical simulations are essential. Figures 4(a)–4(c) summarize the salient features for $\lambda(t)=\lambda_0+\epsilon t$ with $\lambda_0=1$ and $\epsilon=0.001$. For ready comparison, the values corresponding to $\epsilon=0$ have also been shown. As can be seen, the number of level crossings in a given time interval, and its variance, are now enhanced. A qualitative insight into the new law governing the crossings, at least at the level of the mean value, may be gained by appealing once again to the adiabatic approximation now applied at the level of Eq. (4). To the dominant order, this leads to a diffusion equation for $P(x,t)$ with a time-dependent diffusion coefficient, admitting the solution

$$P(x,t) \approx (\ln t)^{-1/2}\exp(-x^2/4\ln t). \tag{43}$$

Inserting this form into Eq. (34) yields the long time behavior

$$\langle N(0;0,T)\rangle \approx T(\ln T)^{-1/2}. \tag{44}$$

The full line joining the data points for the case of the ramp in Fig. 4(a) is a fit to the data provided by the adiabatic approximation above. Once again, the excellent fit corroborates the accuracy of the asymptotic analysis.

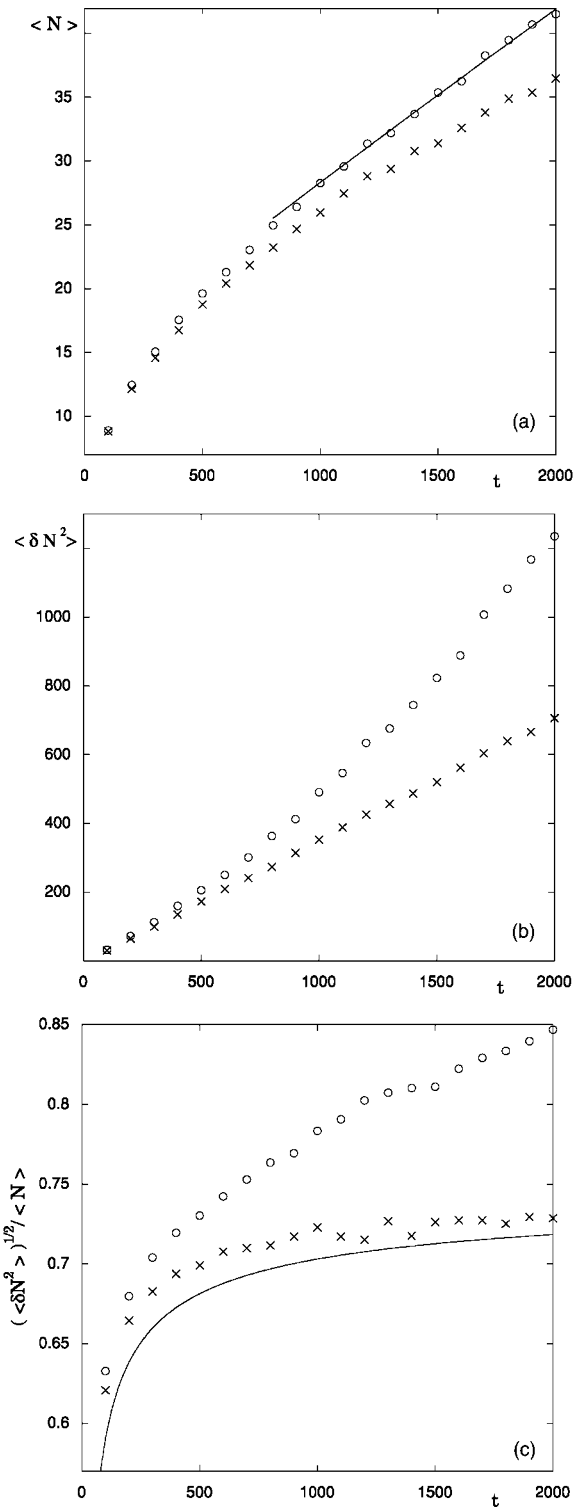

FIG. 4. Mean (a), variance (b), and relative fluctuation (c) of the number of zero crossings for dichotomous diffusion in a time interval from $t=0$ to $t=2000$ time units sampled every 100 time units, as obtained from Eq. (1) with $\epsilon=0$ (crosses) and $\epsilon=0.001$ (empty circles). The full lines in (a) and (c) represent the asymptotic analytical results, Eqs. (44) and (41), respectively. Parameter values as in Fig. 2, but averaging has been performed over 20 000 realizations, with initial conditions randomly chosen between $\xi(0)=+c$ and $\xi(0)=-c$.

IV. GENERAL DICHOTOMOUS FLOW

A. Preliminaries

In this section we consider the case of a general dichotomous flow, in which Eq. (1) is replaced by the stochastic differential equation

$$\dot{x}=f(x)+g(x)\xi(t). \tag{45}$$

The master equations for $P_+(x,t)$ and $P_-(x,t)$ now read

$$\frac{\partial P_+}{\partial t}+\frac{\partial(f_+P_+)}{\partial x}=\lambda(t)(P_- - P_+)$$

and

$$\frac{\partial P_-}{\partial t}+\frac{\partial(f_-P_-)}{\partial x}=\lambda(t)(P_+ - P_-), \tag{46}$$

where

$$f_+(x)=f(x)+cg(x),\quad f_-(x)=f(x)-cg(x). \tag{47}$$

Even in the standard case of a constant switching rate λ, a partial differential equation of finite order can be derived from Eqs. (46) for the total density $P=P_+ + P_-$ only for certain choices [16,17] of $f(x)$ and $g(x)$. However, the corresponding stationary density exists (for constant λ) under fairly general conditions in the ''region of stability'' where $c^2g^2-f^2>0$ or $f_+f_-<0$, i.e., where the flows in the two states of the noise are oppositely directed. This density is given by [8]

$$P^{st}(x)=\pm\frac{\mathcal{N}cg}{c^2g^2-f^2}\exp\left(2\lambda\int\frac{f\,dx}{c^2g^2-f^2}\right) \tag{48}$$

where the + (−) sign applies if $f_+>0, f_-<0$ ($f_+<0, f_->0$). $\mathcal{N}$ is the normalization constant. We shall find it useful to rewrite Eq. (48) in the form [17]

$$P^{st}(x)=\frac{\mathcal{N}}{2}\left(\frac{1}{|f_+|}+\frac{1}{|f_-|}\right)\exp\left(-\lambda\int\frac{dx}{f_+}\right)\times\exp\left(-\lambda\int\frac{dx}{f_-}\right), \tag{49}$$

where the two terms in the sum correspond to $P_+^{st}(x)$ and $P_-^{st}(x)$, respectively. This separation is necessary, for instance, in deriving the level-crossing formula to be presented below, in Eq. (63).

Our primary interest here is in the effects of a time-dependent switching rate $\lambda(t)$, and therefore we shall mainly focus on a nontrivial flow that is, nevertheless, tractable. This is the case of a linear drift, given by $f(x)=-\gamma x$ and $g(x)=1$, i.e.,

$$\dot{x}=-\gamma x+\xi(t), \tag{50}$$

where $\gamma>0$. As the noise $\xi(t)$ switches between its two levels, there is an alternation between flows with stable critical points at c/γ and $-c/\gamma$, respectively. In this case $P(x,t)$ satisfies a second order equation, which is most conveniently written for our present purposes in the form

$$\frac{\partial^2 P}{\partial t^2}=2\gamma\frac{\partial}{\partial x}x\frac{\partial P}{\partial t}-[2\lambda(t)-\gamma]\frac{\partial P}{\partial t}+c^2\frac{\partial^2 P}{\partial x^2}-\gamma^2\frac{\partial}{\partial x}x\frac{\partial(xP)}{\partial x}+\gamma[2\lambda(t)-\gamma]\frac{\partial(xP)}{\partial x}. \quad (51)$$

In the special case of a constant λ, the time-dependent solution $P(x,t)$ of Eq. (51) is in fact known in closed form [16], and is expressible in terms of hypergeometric functions. The corresponding normalized stationary (asymptotic) density $P^{st}(x)$ has a support in the interval bounded by the critical points $\pm c/\gamma$, where it is given by [8]

$$P^{st}(x)=\frac{\gamma\Gamma((\lambda+2\gamma)/2\gamma)}{\sqrt{\pi}c^{-1+2\lambda/\gamma}\Gamma(\lambda/\gamma)}(c^2-\gamma^2x^2)^{-1+\lambda/\gamma}. \quad (52)$$

We shall therefore take x_0 to lie in this interval. Further, for our present purposes, in particular, for deriving and analyzing the equations satisfied by the moments of x, we shall also require that the first derivative dP^{st}/dx be bounded at the end points in x, which requires $\lambda \geqslant 2\gamma$. Hence we restrict ourselves to this range. The changes that arise for smaller values of λ are of interest in their own right [for instance, it is clear from Eq. (52) that there is a qualitative difference [16] in the shape of this density between the cases $\lambda>\gamma$ and $\lambda<\gamma$], but we do not digress to consider this aspect here.

B. Moment equations

From Eq. (51), we find that the moments $m_1=\langle x(t)\rangle$ and $m_2=\langle x^2(t)\rangle$ satisfy the equations

$$\frac{d^2 m_1}{dt^2}+[2\lambda(t)+\gamma]\frac{dm_1}{dt}+2\gamma\lambda(t)m_1=0 \quad (53)$$

and

$$\frac{d^2 m_2}{dt^2}+[2\lambda(t)+3\gamma]\frac{dm_2}{dt}+2\gamma[2\lambda(t)+\gamma]m_2=2c^2. \quad (54)$$

The initial conditions that m_1 and m_2 now satisfy follow from the initial conditions

$$P(x,0)=\delta(x-x_0)$$

and

$$[\partial P/\partial t]_{t=0}=-\frac{\partial}{\partial x}[(c-\gamma x)\delta(x-x_0)], \quad (55)$$

and are given by

$$m_1(0)=x_0, \quad \dot{m}_1(0)=c-\gamma x_0,$$

$$m_2(0)=x_0^2, \quad \dot{m}_2(0)=2x_0(c-\gamma x_0). \quad (56)$$

Again considering first the case of a time-independent λ, we find the solutions

$$m_1(t)=x_0e^{-\gamma t}+\frac{c}{(2\lambda-\gamma)}(e^{-\gamma t}-e^{-2\lambda t}) \quad (57)$$

and

$$m_2(t)=\frac{c^2}{\gamma(2\lambda+\gamma)}+\left\{x_0^2+\frac{c(2\gamma x_0-c)}{\gamma(2\lambda-\gamma)}\right\}e^{-2\gamma t}+\frac{2c}{(2\lambda-\gamma)}\left\{\frac{c}{(2\lambda+\gamma)}-x_0\right\}e^{-(2\lambda+\gamma)t}. \quad (58)$$

These solutions are to be compared with the expressions in Eqs. (9) and (10) which obtain in the case of free dichotomous diffusion, to which they reduce in the limit $\gamma\to 0$. Owing to the presence of a stable drift, the moments now lose their dependence on the initial conditions as $t\to\infty$. The mean value m_1 rises above its initial value x_0 for small t [because of the choice $\xi(0)=+c$ for the initial noise level], and subsequently falls back to zero asymptotically. The variance no longer displays diffusive behavior, but instead approaches a constant, $c^2/\gamma(2\lambda+\gamma)$. This is consistent with the value obtained directly from $P^{st}(x)$ as given by Eq. (52).

When $\lambda(t)$ increases on a ramp, the adiabatic approximation implied by the scaling in Eq. (21) may again be employed to draw conclusions regarding the asymptotic behavior of the moments. This yields, to leading order,

$$m_1(t)\approx 0, \quad m_2(t)\approx\frac{c^2}{\gamma[2(\lambda_0+\epsilon t)+\gamma]}. \quad (59)$$

Figures 5(a) and 5(b) show, respectively, the behavior of the mean and the variance for a set of values of the ramp parameter ϵ ranging from 0 to 0.5. Here λ_0 and c have been set equal to unity, while $\gamma=0.25$ (so that $\lambda>2\gamma$). A more detailed look at the variance induced by the ramp in the long time regime is provided by Fig. 5(c), where the full lines correspond to a fit by the expression in the second of Eqs. (59). The agreement with the results of simulations is, again, quite satisfactory.

C. Level crossings

1. General formulas

Starting with the general formula of Eq. (32), an expression may be derived for the mean number of threshold crossings for the arbitrary dichotomous flow of Eq. (45) as follows. Using the flow equation to eliminate the velocity variable in favor of ξ, we have

$$P(x,\dot{x},t)=\langle\delta(\dot{x}-f(x)-g(x)\xi)\rangle \quad (60)$$

where the averaging is over the sample space of ξ. This leads to

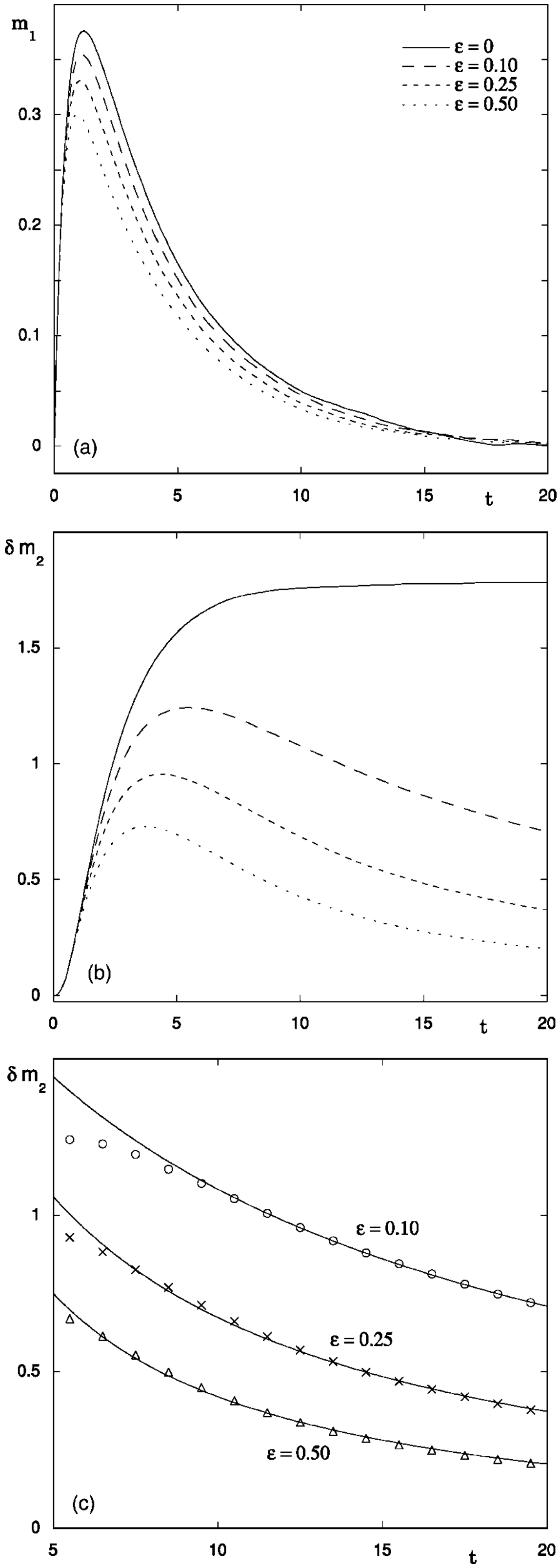

FIG. 5. As in Fig. 2, but in the presence of a linear drift term [Eq. (50)] and for different values of the ramp parameter ϵ. In (c) the numerically obtained values of the second moment (empty circles, crosses, and triangles) are compared with the asymptotic analytical result (full line), Eq. (59). Parameter values as in Fig. 2, with $\gamma = 0.25$; number of realizations 50 000.

$$P(x,\dot{x},t) = \delta(\dot{x} - f_+(x))P_+(x,t) + \delta(\dot{x} - f_-(x))P_-(x,t). \tag{61}$$

Formal expressions for the mean number of upcrossings and downcrossings of a given level x_{th} can now be written down. For the mean total number of crossings in the time interval from 0 to T, we obtain the simple formula

$$\langle N(x_{th};0,T)\rangle = \int_0^T dt[|f_+(x_{th})|P_+(x_{th},t) + |f_-(x_{th})|P_-(x_{th},t)]. \tag{62}$$

The integrand in Eq. (62) can be identified with an instantaneous mean rate $r(x_{th},t)$ of crossings of the level concerned. If a stationary distribution exists, as in Eq. (48) or Eq. (49) above, then the mean rate of crossings asymptotically approaches the stationary value

$$r^{st}(x_{th},t) = \mathcal{N} \exp\left(2\lambda \int^{x_{th}} \frac{f\,dx}{c^2 g^2 - f^2}\right). \tag{63}$$

2. Level crossings in linear dichotomous flow

Turning once again to the linear dichotomous flow of Eq. (50), the known solutions for $P_\pm(x,t)$ may be used in Eq. (62) to write down $\langle N(x_{th};0,T)\rangle$ explicitly, in principle. As the solutions $P_\pm(x,t)$ involve very lengthy expressions, we do not do this here.

For the case of a constant λ, the asymptotic value of the mean rate of crossings can be written down from Eq. (63). We find

$$r^{st}(x_{th},t) = \frac{\gamma\Gamma((\lambda+2\gamma)/2\gamma)}{\sqrt{\pi}c^{2\lambda/\gamma}\Gamma(\lambda/\gamma)}(c^2 - \gamma^2 x_{th}^2)^{\lambda/\gamma}. \tag{64}$$

For zero crossings, this simplifies further to

$$r^{st}(0,t) = \frac{\gamma\Gamma((\lambda+2\gamma)/2\gamma)}{\sqrt{\pi}\Gamma(\lambda/\gamma)}. \tag{65}$$

For $\lambda(t)$ varying on the ramp, analytic expressions for $P_\pm(x,t)$ are not available. However, we may use the same adiabatic approximation as in the preceding sections to arrive at the leading asymptotic behavior of various quantities. We thus find, for sufficiently large t,

$$P(x,t) \approx \frac{\gamma\Gamma((\lambda(t)+2\gamma)/2\gamma)}{\sqrt{\pi}c^{-1+2\lambda(t)/\gamma}\Gamma(\lambda(t)/\gamma)} \times (c^2 - \gamma^2 x^2)^{-1+\lambda(t)/\gamma}, \tag{66}$$

where $\lambda(t) = \lambda_0 + \epsilon t$. This is used in the analysis of the numerical results presented in Figs. 6 and 7, where $\lambda_0 = 1, c = 1$, and $\gamma = 0.25$, as in the preceding subsection.

In numerical simulations, it is more convenient to work with the mean $\langle N(x_{th};0,T)\rangle$ itself, rather than the rate $r(x_{th},t)$. Figure 6(a) compares the mean number of zero crossings up to time t for $\epsilon=0$ and $\epsilon=0.001$, respectively, while Fig. 6(b) does the same for the corresponding variances. Although there appear to be significant differences between the two cases, the relative fluctuation $\langle \delta N^2(0;0,t)\rangle^{1/2}/\langle N(0;0,t)\rangle$ is practically the same in both cases, as demonstrated in Fig. 6(c). Figures 7(a)–7(c) are the counterparts of Figs. 6(a)–6(c) for a much larger value of ϵ, namely, $\epsilon=0.1$. The full line running along the data points corresponding to the ramp in both Fig. 6(a) and Fig. 7(a) is a fit to the data using the adiabatic approximation, which yields

$$\langle N(0;0,t)\rangle \approx \frac{\gamma}{\sqrt{\pi}}\int_0^t dt' \frac{\Gamma((\lambda_0+\epsilon t'+2\gamma)/2\gamma)}{\Gamma((\lambda_0+\epsilon t')/\gamma)}. \quad (67)$$

Once again, we see that the fit is very good indeed.

V. THE CASE OF MULTILEVEL NOISE

A. The model

Finally, we extend some of the results obtained in the foregoing to the case of a multilevel generalization of dichotomous noise, but one which preserves nevertheless the features of discreteness, non-Gaussianity, and a finite correlation time that the latter possesses. Our motivation is to get an idea of the role played by the dimensionality of the state space of the discrete noise. In practical modeling, one may well have a discrete *set* of values for the random forcing that comprises more than just two levels.

The specific model we shall consider is a generalization of dichotomous diffusion, given by the flow $\dot{x}=\xi(t)$, where the velocity $\xi(t)$ is a q-state Markov process that takes the values $c_1,\ldots,c_q$, and q is a positive integer $\geqslant 2$. As before, the jumps in $\xi(t)$ are assumed to occur at random instants of time, at a mean transition rate λ (which is time dependent in general). For simplicity, we also assume that from any level c_i of the noise, a jump may occur to *any* other level $c_j (j \neq i)$ with equal probability $[=(q-1)^{-1}]$. This assumption may be relaxed or modified, to lead to various other models—for instance, transitions restricted to $c_i \to c_{i\pm 1}$ lead to a generalized model of Taylor dispersion [11], and so on.

Denoting by $P_i(t)$ the probability that $\xi(t)=c_i$, the statistics of the noise is given by the master equation

$$\dot{P}_i(t) = -(q-1)\lambda(t)P_i(t) + \lambda(t)\sum_{j\neq i}^{q} P_j(t). \quad (68)$$

For an arbitrary initial level $\xi(0)=c_k$ of the noise, the solution to Eq. (68) is given by

$$P_k(t)=q^{-1}+(1-q^{-1})e^{-q\int_0^t \lambda(t')dt'}$$

and

$$P_i(t)=q^{-1}(1-e^{-q\int_0^t \lambda(t')dt'}), \quad i\neq k. \quad (69)$$

This is a generalization of Eq. (20). The lifetime distribution of any state of the noise is now

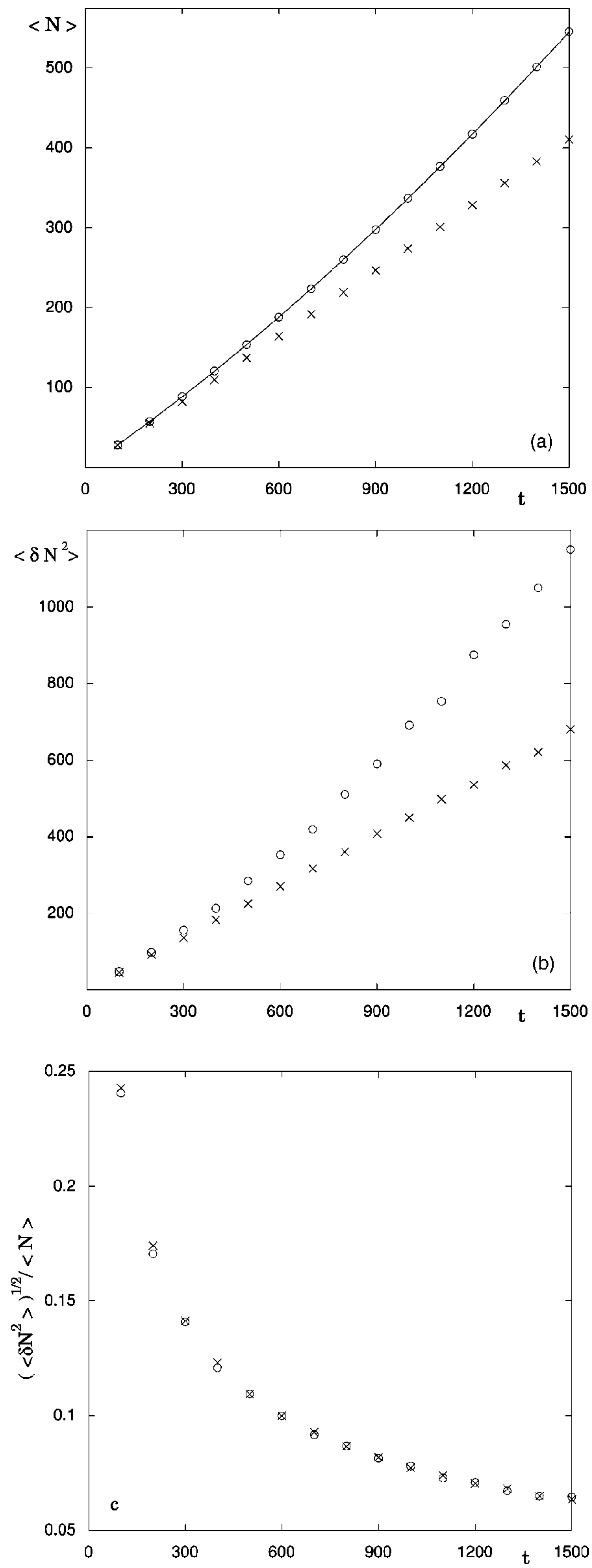

FIG. 6. As in Fig. 4, but in the presence of a linear drift, Eq. (50). The full line in (a) represents the analytical result, Eq. (67). Parameter values as in Fig. 5. Number of realizations 10 000; initial conditions $x_0=0$, $\xi_0=+c$.

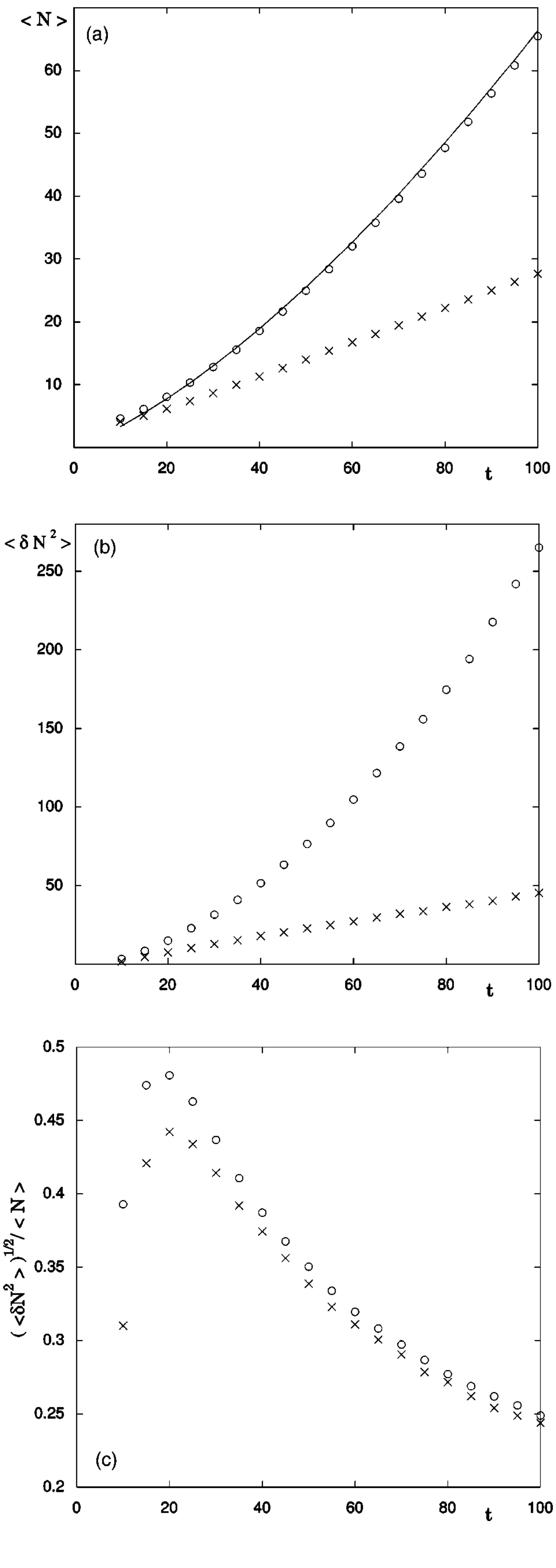

FIG. 7. As in Fig. 6, but with $\epsilon=0$ (crosses) and $\epsilon=0.1$ (empty circles). Number of realizations 10^5.

$$\pi(t)=\exp\left(-(q-1)\int_0^t \lambda(t')dt'\right), \tag{70}$$

where the extra factor of $(q-1)$ in the exponent must be noted.

The master equation for the probability densities $\{P_i(x,t)\}$ of the diffusion process reads

$$\frac{\partial}{\partial t}P_i(x,t)+c_i\frac{\partial}{\partial x}P_i(x,t)=-(q-1)\lambda(t)P_i(x,t)+\lambda(t)\sum_{j\neq i}^{q} P_j(x,t). \tag{71}$$

For the first moment $m_1=\langle x(t)\rangle$ this yields the equation

$$\frac{dm_1}{dt}=\sum_{i=1}^{q} c_iP_i(t), \tag{72}$$

where $P_i(t)$ is given by Eq. (69). With the initial condition $m_1(0)=x_0$, the solution of Eq. (72) is

$$m_1(t)=x_0+\langle c\rangle t+(c_k-\langle c\rangle)\int_0^t dt' e^{-q\int_0^{t'}\lambda(\tau)d\tau}, \tag{73}$$

where $\langle c\rangle=q^{-1}\Sigma_1^q c_i$ is the mean drift velocity. We shall henceforth take $\langle c\rangle=0$, to eliminate this trivial drift.

The calculation of the second moment is more elaborate in this general case, since the equation satisfied by $m_2(t)$ is

$$\frac{dm_2}{dt}=2\sum_{i=1}^{q} c_im_{1i}(t), \tag{74}$$

where $m_{1i}(t)$ is the "partial" first moment $\int xP_i(x,t)dx$. These quantities have therefore to be determined first. The formal solution for $m_2(t)$ that finally results is quite lengthy, and we do not give it here.

B. Time-independent λ

For constant λ, we obtain

$$m_1(t)=x_0+\frac{c_k}{q\lambda}(1-e^{-q\lambda t}), \tag{75}$$

which is the generalization of Eq. (9). (We recall that c_k is the initial velocity state.) As $t\to\infty$, this saturates to the value $x_0+c_k/(q\lambda)$. Again, this is corroborated by numerical simulation. Figure 8(a) shows the behavior of the mean in the case $q=4, c_i=\{-2,-1,+1,+2\}$, and $c_k=1$. The rate λ_0 has been set equal to $\frac{1}{3}$ in order to compensate for the factor $(q-1)$ in the exponent in Eq. (70), thereby facilitating ready comparison of numerical values with those obtained in the preceding sections in the case of dichotomous diffusion. The curve corresponding to $\epsilon=0$ in Fig. 8(a) saturates to a value 0.75, in agreement with the theoretical prediction.

The variance of x in the case of a constant λ is found to be

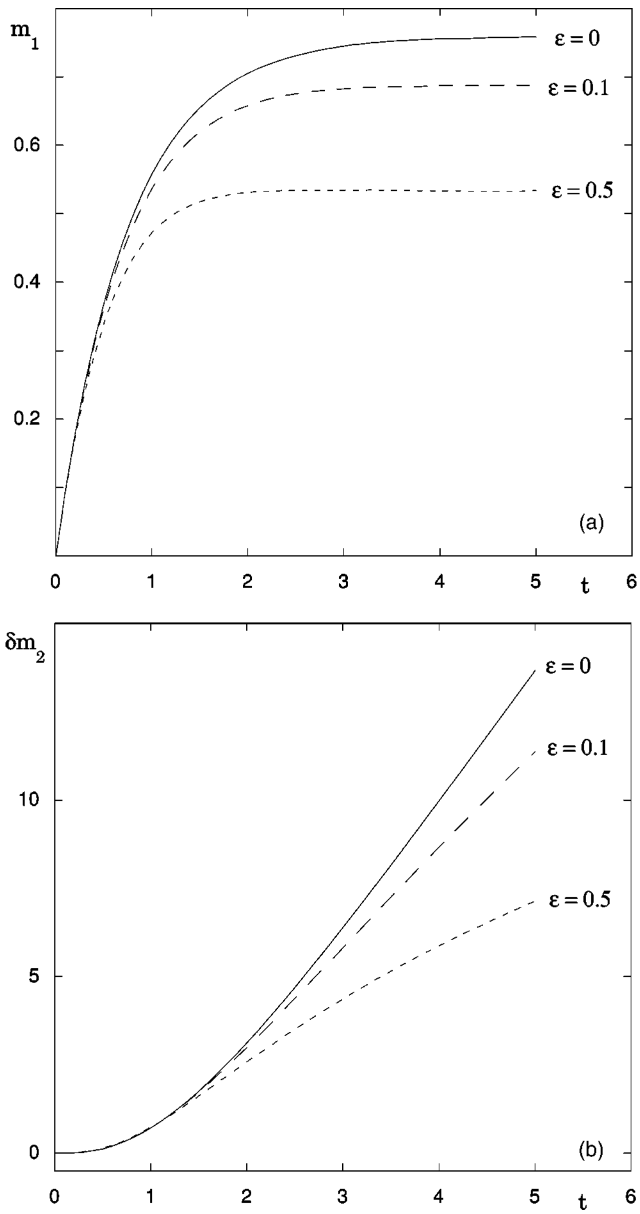

FIG. 8. As in Fig. 2, but for a four-level noise ($q=4$), with $c_i=\{-2,-1,+1,+2\}$ and $\lambda_0=1/3$.

$$\delta m_2(t)=\frac{2\langle c^2\rangle}{q\lambda}t-\frac{(4\langle c^2\rangle-c_k^2)}{(q\lambda)^2}+\frac{4\langle c^2\rangle}{(q\lambda)^2}e^{-q\lambda t}+\frac{2(\langle c^2\rangle-c_k^2)}{q\lambda}te^{-q\lambda t}-\frac{c_k^2}{(q\lambda)^2}e^{-2q\lambda t}, \quad (76)$$

where $\langle c^2\rangle=q^{-1}\Sigma_i c_i^2$. The effective diffusion constant in this generalization of dichotomous diffusion is therefore $D=\langle c^2\rangle/(q\lambda)$, in complete agreement with the results of simulation, as seen from the curve corresponding to $\epsilon=0$ in Fig. 8(b). If $\langle c\rangle\neq 0$, the diffusion constant in this model is $(\langle c^2\rangle-\langle c\rangle^2)/(q\lambda)$.

C. λ increasing on a ramp

For λ increasing with time on a ramp as in Eq. (15), we find

$$m_1(t)=x_0+c_k\sqrt{\frac{\pi}{2q\epsilon}}e^{q\lambda_0^2/2\epsilon}\times\left[\operatorname{erf}\left(t\sqrt{\frac{q\epsilon}{2}}+\lambda_0\sqrt{\frac{q}{2\epsilon}}\right)-\operatorname{erf}\left(\lambda_0\sqrt{\frac{q}{2\epsilon}}\right)\right]. \quad (77)$$

This is the generalization of Eq. (16) to our q-state model. For $t\gg 1/\sqrt{\epsilon}$, we now get

$$m_1(t)\approx x_0+\frac{c_k}{2\lambda_0}-\frac{c_k\epsilon}{q^2\lambda_0^3}\cdots. \quad (78)$$

This gives an idea of the relative roles played by ϵ and q in the stabilization of the mean (in the sense already explained in Sec. II C) under the action of the ramp. The numerical results are again in agreement with the theoretical predictions, as seen in Fig. 8(a). Finally, in Fig. 8(b) for the variance, the curves corresponding to nonzero values of ϵ, now drawn entirely on the basis of the numerical simulation, indicate the expected trend toward subdiffusive behavior for asymptotically long times.

Regarding level-crossing statistics in the multilevel model under discussion, we note that a formal expression for the mean number of crossings of a threshold x_{th} may easily be written down as a generalization of that obtained in the dichotomous case. In fact, for the multilevel *flow* given by $\dot{x}=f(x)+g(x)\xi(t)$ where ξ is the q-state Markov noise defined in this section, the formula in Eq. (62) generalizes directly to

$$\langle N(x_{th};0,T)\rangle=\int_0^T dt\sum_{i=1}^{q}|f_i(x_{th})|P_i(x_{th},t), \quad (79)$$

where $f_i(x)=f(x)+c_i g(x)$. However, we do not pursue this aspect any further here, as no analytic solution for $P(x,t)$ (in the case of constant λ) is available as a reference and for comparison with the results of simulation with constant as well as time-varying λ.

VI. CONCLUSIONS

The principal result of this work has been to show that a systematic variation in time of a control parameter, in a dynamical system driven by dichotomous or multilevel noise, may introduce *qualitative* changes in its statistical properties relative to the unforced case. These changes already appear in the case where the deterministic dynamics in the absence of noise is very simple. These may involve, for instance, the switching from diffusive to subdiffusive, or, on the contrary, ballistic behavior of the variance; or the enhancement of the crossings of some prescribed threshold by the stochastic trajectory.

Our analysis of the level-crossing dynamics has also revealed some rather unexpected features that are already exhibited in the case of a fixed control parameter. It has in fact been shown that the fluctuations of the number of crossings about its mean value are comparable to the mean itself. This property should have important repercussions in the interpretation and prediction of extremes, especially in connection with their next likely recurrence, a problem of the utmost importance in environmental science and engineering.

The analysis carried out in the present paper can be extended in several directions. One such would be to the case of nonlinear dichotomous flows, particularly flows that admit instabilities and multiple states in the deterministic limit. The interference of these instabilities with the stabilizing or destabilizing trends induced by the time variation of the control parameter could then lead to different modes of behavior that would be worth exploring. Another possible extension is the investigation of the role of an external periodic forcing superimposed on the dichotomous noise. The main question here is the possibility of an enhancement of the response of the system via a mechanism of the stochastic resonance type.

Finally, the investigation of dynamical systems forced by other forms of noise and subject to systematic time variation of their control parameters could reveal whether ‘‘universality classes’’ can be identified, as far as the moment and level-crossing dynamics are concerned. In particular, the precise roles of the discreteness, the non-Gaussian character, and the correlation time of the noise would be worth clarifying in greater detail.

From the standpoint of applications, we believe that the results reported here provide useful tools for approaching prediction-related problems of considerable concern in the context of atmospheric science and also economics and finance, among others.

ACKNOWLEDGMENTS

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